BE 150 Spring 2018
Homework #9
Due in Justin's mailbox by 11AM, June 6, 2018.

Problem 9.1 (Turing patterns with expanders (70 pts)).
In this problem, we will explore scaling of Turing patterns, described in Werner, et al., *PRL*, **114**, 138101, 2015, which you can download here. It will be helpful for you to refer to that paper in working this problem, but bear in mind that here we are using the same notation we have used throughout the course. I.e., γ_A describes a degradation, as opposed to β_A describing a degradation as it does in the paper.

As discussed in lecture, Turing patterns tend to have a wavelength that is independent of the size of the system, which means that they do not scale. Werner and coworkers propose a model for interactions of an expander with the classical Turing activator (A)-inhibitor (B) components. This model is shown in Fig. 3a of the paper.

a) The authors use the following set of PDEs for the Turing system without the expander.

$$\frac{\partial A}{\partial t} = D_A \frac{\partial^2 A}{\partial x^2} + \beta_A \frac{A^n}{A^n + B^n} - \gamma_A A \tag{9.1}$$

$$\frac{\partial B}{\partial t} = D_B \frac{\partial^2 B}{\partial x^2} + \beta_B \frac{A^n}{A^n + B^n} - \gamma_B B. \tag{9.2}$$

i) They seem to have chosen a peculiar functional form for the mutual regulation of A and B, $A^n/(A^n + B^n)$. Describe in words how this captures the components of the diagram for the Turing system (inside the dotted box of Fig. 3a of the paper).

ii) Show that we can nondimensionalize this system to be

$$\frac{\partial a}{\partial t} = \frac{\partial^2 a}{\partial x^2} + \frac{a^n}{a^n + b^n} - a \tag{9.3}$$

$$\frac{\partial b}{\partial t} = d_b \frac{\partial^2 b}{\partial x^2} + \beta_b \frac{a^n}{a^n + b^n} - \gamma_b b, \tag{9.4}$$

where x and t are now dimensionless. Importantly, x has been nondimensionalized with the characteristic length scale $\lambda_A = \sqrt{D_A/\gamma_A}$.

iii) Find the unique homogeneous steady state of this system. By homogeneous steady state, we mean that the concentration profiles of a and b are uniform in space and the time derivatives vanish.

iv) Starting from a small perturbation of this steady state, numerically solve the coupled PDEs. Use no-flux boundary conditions. Plot the resulting steady state concentration profiles for a and b. Do this using the following parameters: $d_b = 30$, $\beta_b = 4$, $\gamma_b = 2$, and $n = 5$. Do this five times, once each for the total dimensionless length of the system being 5, 10, 20, 40, and 80. Comment on the pertinence of these results with respect to scaling.

b) We will now consider the system in the presence of an expander, E. The new dimensional equations are

$$\frac{\partial A}{\partial t} = D_A \frac{\partial^2 A}{\partial x^2} + \beta_A \frac{A^n}{A^n + B^n} - \kappa_A E A \tag{9.5}$$

$$\frac{\partial B}{\partial t} = D_B \frac{\partial^2 B}{\partial x^2} + \beta_B \frac{A^n}{A^n + B^n} - \kappa_B E B \tag{9.6}$$

$$\frac{\partial E}{\partial t} = D_E \frac{\partial^2 E}{\partial x^2} + \beta_E - \kappa_E E B. \tag{9.7}$$

i) Give a brief description on how this relates to the schematic in Fig. 3a of the Werner, et al. paper and comment on any approximations that are being made.

ii) Show that these equations may be nondimensionalized to read

$$\frac{\partial a}{\partial t} = \frac{\partial^2 a}{\partial x^2} + \frac{a^n}{a^n + b^n} - ea \tag{9.8}$$

$$\frac{\partial b}{\partial t} = d_b \frac{\partial^2 b}{\partial x^2} + \beta_b \frac{a^n}{a^n + b^n} - \kappa_b e b, \tag{9.9}$$

$$\frac{\partial e}{\partial t} = d_e \frac{\partial^2 e}{\partial x^2} + \beta_e - \kappa_e e b, \tag{9.10}$$

where again x and t are now dimensionless. Importantly, note that this nondimensionalization has a different length scale than the original system. We now have $\lambda_0 = \sqrt{D_A/\sqrt{\kappa_A \beta_A}}$.

iii) There is no homogeneous steady state for this system. For our initial "steady state" when doing the numerical calculations in part (iv), we will take $e_0 = 1$ and solve for the values of a and b such that $\mathrm{d}a/\mathrm{d}t = \mathrm{d}b/\mathrm{d}t = 0$. Nevermind that for this "steady state," $\mathrm{d}e/\mathrm{d}t \neq 0$.

iv) Again, starting from a small perturbation of this "steady state," numerically solve the coupled PDEs using no-flux boundary conditions. Plot the resulting steady state concentration profiles for a and b. Do this using the same parameters as in part (a-iv), with additional parameters $d_e = 10$, $\beta_e = 0.4$, $\kappa_b = 2$, and $\kappa_e = 2$. Again, do this five times, once each for the total dimensionless length of the system being 5, 10, 20, 40, and 80. (You can do it for more lengths if you like.) Comment on the pertinence of these results with respect to scaling.

c) Comment qualitatively on how the expander works to scale the Turing patterns. By what other means (other than mutual inhibition of B and inhibition of A) might an expander operate? Remember, these models are postulates of what *might* be happening in developing organisms, so it is useful to dream up alternatives.

Problem 9.2 (Course feedback (30 pts)).
We are always looking to improve our course. Please provide feedback for the following questions. And thanks for a great term!

a) We are considering writing an electronic book (freely available) for the material in this course. We would then use that book as primary reading material, and we would not spend much time at all in class lecturing. Rather, we will expect students to have completed reading ahead of class and we will have discussions and work out some analyses (and maybe even designs) of circuits in class. What do you think of this proposition?

b) Were there any design principles you found particularly difficult to understand? What do you think was the source of this lack of clarity?

c) Were there any techniques you learned in class that were difficult to understand and/or implement? Please comment on what you found troublesome.

d) Please provide any other suggestions you may have for the course.